

Hyperfinite-Dimensional Representations of Canonical Commutation Relation

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Abstract

This paper presents some methods of representing canonical commutation relations in terms of hyperfinite-dimensional matrices, which are constructed by nonstandard analysis. The first method uses representations of a nonstandard extension of finite Heisenberg group, called hyperfinite Heisenberg group. The second is based on hyperfinite-dimensional representations of $so(3)$. Then, the cases of infinite degree of freedom are argued in terms of the algebra of hyperfinite parafermi oscillators, which is mathematically equivalent to a hyperfinite-dimensional representation of $so(n)$.

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I. INTRODUCTION

In the field of mathematical foundations of quantum physics, especially of infinite degree of freedom, it is often argued that we need any extensions of separable Hilbert space as the fundamental mathematical tools, such as the linear space of generalized functions.

In recent years, another alternative, *nonstandard-analytical extension* method, is developed by some authors^{1,2,3,4}. The present paper adopts an approach in use of hyperfinite-dimensional linear spaces, constructed by non-standard analysis^{5,6} (see Appendix to find the basic definitions).

This approach also is closely related with that of *finite-dimensional quantum physics*^{7,8,9,10}; nonstandard extensions of finite-dimensional models in quantum physics turn out to be the powerful tools for the approach.

The purpose of the present paper is representing canonical commutation relation (CCR) $[Q, P] = i$ (let us take the natural unit $\hbar = 1$), or equivalently, $[a, a^\dagger] = 1$ where $a = (Q + iP)/\sqrt{2}$, in a hyperfinite-dimensional linear space. Let ν be an infinite hypernatural number, and Q and P internal $\nu \times \nu$ hermitian matrices. Q and P act on the ν -dimensional internal linear space ${}^*\mathbf{C}^\nu$ as internal linear operators. No standard finite-dimensional matrices satisfy CCR. By the transfer principle in nonstandard analysis, the fact also holds for any internal hyperfinite-dimensional matrices, so we have $[Q, P] \neq i$. Thus we must arrange the standard CCR for hyperfinite-dimensional matrices. First we consider the following weaker version of CCR: $[Q, P] \approx i$, that is, $\|[Q, P] - i\|_\infty \approx 0$, where $\|A\|_\infty$ is given by

$$\|A\|_\infty = \sup\{\|A\xi\| : \|\xi\| \leq 1, \xi \in {}^*\mathbf{C}^\nu\}$$

This is equivalent to the condition that for any $\xi \in {}^*\mathbf{C}^\nu$ with finite norm, $[Q, P]\xi \approx i\xi$ (i.e., $\|([Q, P] - i)\xi\| \approx 0$ where $\|\cdot\|$ is the norm on ${}^*\mathbf{C}^\nu$ defined by $\|(\xi_1, \dots, \xi_\nu)\| = \sqrt{\xi_1^2 + \dots + \xi_\nu^2}$). We see that the condition never holds as follows. Suppose the condition holds. Then any eigenvalue λ of $[Q, P]$ satisfies $\lambda \approx i$. This contradicts $\text{Tr}([Q, P]) = 0$. Now, let us define the still weaker notion of representation of CCR by limiting the domain of the operators Q and P .

Definition 1.1: Let Q_k, P_k be $\nu \times \nu$ internal hermitian matrices ($k = 1, 2, \dots < \infty$), and S be an external subspace of ${}^*\mathbf{C}^\nu$ which consists only of the vectors with finite norm. The set of the triples (Q_k, P_k, S) (or of pairs (a_k, S) where $a_k = (Q_k + iP_k)/\sqrt{2}$) is called a *quasi-representation of CCR* if

$$[Q_k, P_l]\xi \approx i\delta_{kl}\xi, \quad [Q_k, Q_l]\xi \approx [P_k, P_l]\xi \approx 0$$

for any $\xi \in S$ and $k, l = 1, 2, \dots < \infty$. A quasi-representation of CCR (Q_k, P_k, S) is called a *representation of CCR* if S is invariant respect to Q_k, P_k (i.e., $Q_k S, P_k S \subseteq S$).

Hereafter, we will use the following notations.

- (1) $*$ is the mapping of nonstandard extension.
- (2) $\mathbf{N}^*(\mathbf{N})$ is the set of natural (hypernatural) numbers.
- (3) $\mathbf{Z}^*(\mathbf{Z})$ is the set of integers (hyperintegers).
- (4) $\mathbf{R}^*(\mathbf{R})$ is the set of real (hyperreal) numbers.
- (5) $\mathbf{C}^*(\mathbf{C})$ is the set of complex (hypercomplex) numbers.

II. CANONICAL PAIR OF UNITARY MATRICES

Definition 2.1: Let $M^*(\mathbf{C}, \nu)$ denote the set of internal $\nu \times \nu$ matrices ($\nu \in \mathbf{N}^*$). Unitary matrices $U, V \in M^*(\mathbf{C}, \nu)$ are called a *canonical pair* if $U^k, V^k \neq 1$ ($k = 1, \dots, \nu - 1$), $U^\nu = V^\nu = 1$ and $UV = e^{2\pi i/\nu} VU$.

Let $|u_0\rangle$ be a normalized eigenvector of U with eigenvalue c_0 .

Lemma 2.1: For any $n \in \mathbf{N}^*$, $V^n|u_0\rangle$ is an eigenvector of U , and its eigenvalue is $e^{2\pi ni/\nu} c_0$.

Proof: Induction for n . $UV^{n+1}|u_0\rangle = UVV^n|u_0\rangle = e^{2\pi i/\nu} VUV^n|u_0\rangle = e^{2\pi i/\nu} V e^{2\pi ni/\nu} c_0 V^n|u_0\rangle = c_0 e^{2\pi(n+1)i/\nu} V^{n+1}|u_0\rangle$. Q.E.D.

From the definition, we find that all the eigenvalues of U are $e^{2\pi i/\nu}, e^{4\pi i/\nu}, \dots, e^{2\pi(\nu-1)i/\nu} = 1$. Let us assume $c_0 = 1$.

Lemma 2.2: Define $|u_k\rangle$ and $|v_n\rangle$ by $|u_k\rangle = V^k|u_0\rangle$ and $|v_n\rangle = \frac{1}{\sqrt{\nu}} \sum_{k=0}^{\nu-1} e^{2\pi nki/\nu} |u_k\rangle$ ($k, n \in \mathbf{N}^*$). $|v_n\rangle$ is an eigenvector of V , and its eigenvalue is $e^{-2\pi ni/\nu}$.

Proof: $V|v_n\rangle = \frac{1}{\sqrt{\nu}} \sum_{k=0}^{\nu-1} e^{2\pi nki/\nu} V^{k+1}|u_0\rangle = e^{-2\pi ni/\nu} \frac{1}{\sqrt{\nu}} \sum_{k=0}^{\nu-1} e^{2\pi(n+1)ki/\nu} V^k|u_0\rangle = e^{-2\pi ni/\nu} |v_n\rangle$. Q.E.D.

From these lemmas, we see that $|u_k\rangle$ and $|v_k\rangle$ ($k = 0, \dots, \nu - 1$) are complete orthonormal systems of \mathbf{C}^ν . Thus we have

$$V = \sum_{k=0}^{\nu-1} |u_{k+1}\rangle \langle u_k| = \sum_{k=0}^{\nu-1} e^{-2\pi ki/\nu} |v_n\rangle \langle v_n|$$

$$U = \sum_{k=0}^{\nu-1} |v_{k+1}\rangle \langle v_k| = \sum_{k=0}^{\nu-1} e^{2\pi ki/\nu} |u_n\rangle \langle u_n|$$

The following proposition follows from these consequences.

Proposition 2.3: The canonical pair of $\nu \times \nu$ unitary matrices is unique in the sense of unitary equivalence, that is, if (U, V) and (U', V') are two such canonical pairs, then there is a unitary matrix W such that $U' = WUW^\dagger, V' = WVW^\dagger$.

Definition 2.2: Let $A \in M^*(\mathbf{C}, \nu)$. $|a\rangle \in \mathbf{C}^\nu$ is called *approximately invariant* vector of A iff $\langle a|a\rangle < \infty$ and $A|a\rangle \approx |a\rangle$.

Let $I(A)$ denote the set of all the approximately invariant vectors of A . $I(A)$ is an external subspace of ${}^*\mathbf{C}^\nu$.

Theorem 2.4: If $\nu > \infty$, then $I(U) \cap I(V)$ is an invariant subspace of U and V , and is infinite-dimensional (i.e., it has an infinite orthonormal system), and

$$\frac{\nu}{2\pi mn}[U^m, V^n]|a\rangle \approx i|a\rangle.$$

holds for any $|a\rangle \in I(U) \cap I(V)$ and $m, n \in \mathbf{Z}$.

Proof: $I(U) \cap I(V)$ is clearly an invariant subspace. Suppose $l = 0, 1, \dots < \infty$, $\mu > \infty$ and $\mu/\nu \approx 0$. Define $|l\rangle$ by

$$|l\rangle = \frac{1}{\sqrt{\mu}} \sum_{k=l\mu}^{(l+1)\mu-1} |u_k\rangle$$

$\{|0\rangle, |1\rangle, \dots\}$ is an orthonormal system of $I(U) \cap I(V)$. In fact, $\|V|l\rangle - |l\rangle\|^2 = (1/\mu)\| |u_{(l+1)\mu}\rangle - |u_{l\mu}\rangle \|^2 \approx 0$, and $\|U|l\rangle - |l\rangle\|^2 = (1/\mu) \sum_{k=l\mu}^{(l+1)\mu-1} |e^{2\pi\mu i/\nu} - 1|^2 \approx 0$. We have also $(\nu/2\pi mn)(U^m V^n - V^n U^m)|a\rangle = \nu/2\pi mn(e^{2\pi m n i/\nu} - 1)V^n U^m|a\rangle \approx iV^n U^m|a\rangle \approx i|a\rangle$. Q.E.D.

Corollary: Suppose $m, n \in \mathbf{N}$. Let $P^{(m)}, Q^{(n)}$ be hermitian matrices defined by

$$P^{(m)} = \frac{i}{m} \sqrt{\frac{\nu}{8\pi}} (U^m - U^{-m}), \quad Q^{(n)} = \frac{i}{n} \sqrt{\frac{\nu}{8\pi}} (V^n - V^{-n})$$

If $|a\rangle \in I(U) \cap I(V)$, then

$$[Q^{(n)}, P^{(m)}]|a\rangle \approx i|a\rangle,$$

and hence, $(Q^{(n)}, P^{(m)}, I(U) \cap I(V))$ is a quasi-representation of CCR for each m and n .

We will give some other properties of a canonical pair of unitary matrices.

Definition 2.3: Let $K \in \mathbf{N}$. Let \mathbf{K} be the ring of residue classes of \mathbf{Z} modulo K , i.e., $\mathbf{K} = \mathbf{Z}/K\mathbf{Z}$. For $k, k' \in \mathbf{K}$, $k \oplus k'$ stands for the sum of k and k' in \mathbf{K} , $k \otimes k'$ the product of k and k' , and $\ominus k$ the minus k . The *finite Heisenberg group* based on \mathbf{K} is the group H_K with the underlying set $\mathbf{K} \times \mathbf{K} \times \mathbf{K}$ whose group operation is

$$(k, l, m)(k', l', m') = (k \oplus k', l \oplus l', m \oplus m' \oplus (k \otimes l')).$$

The notion of *Hyperfinitesimal Heisenberg group* is the nonstandard (internal) extension of that of finite Heisenberg group; its definition is given by substituting ${}^*\mathbf{N}$ and ${}^*\mathbf{Z}$ for \mathbf{N} and \mathbf{Z} in the above definition, respectively.

Suppose $\nu > \infty$. The canonical pair U, V generates an internal group G . We see

$$G = \{e^{2\pi i m/\nu} V^l U^k \mid k, l, m = 0, 1, \dots, \nu - 1\}$$

The following property is easily seen.

Proposition 2.5: Define the mapping $\pi : H_\nu \rightarrow M^*(\mathbf{C}, \nu)$ by $\pi(k, l, m) = e^{2\pi i m/\nu} V^l U^k$. π is an internal irreducible unitary representation of the hyperfinite Heisenberg group H_ν .

Consider \mathbf{R}^3 with coordinates (p, q, t) . We make \mathbf{R}^3 into a locally compact group with group law

$$(p, q, t)(p', q', t') = (p + p', q + q', t + t' + pq').$$

We call this group the *Heisenberg group* and denote it by H . The *Schödinger representation* of H is the strong continuous unitary representation ρ of H on $L^2(\mathbf{R})$ defined by

$$(\rho(p, q, t)f)(x) = e^{2\pi i(t+qx)} f(x + p)$$

for all $f \in L^2(\mathbf{R})$. This is a realization of Weyl's commutation relation $U(q)V(p) = e^{iqp}V(p)U(q)$ where $U(q)$ and $V(p)$ are one-parameter unitary groups. From the representation of H_ν , Ojima and Ozawa¹ constructed a representation of H in a hyperfinite-dimensional Hilbert space which is unitary equivalent to the Schödinger representation of H . By using this representation, they found a nonstandard-analytical proof of noncommutative Parseval's identity

$$\int \int_{\mathbf{R}^2} \langle \psi_1 | \rho(p, q, 0) | \phi_1 \rangle \langle \phi_2 | \rho(p, q, 0)^{-1} | \psi_2 \rangle dp dq = \langle \phi_2 | \phi_1 \rangle \langle \psi_1 | \psi_2 \rangle,$$

for all $\psi_i, \phi_i \in L^2(\mathbf{R})$ ($i = 1, 2$).

III. $SO(3)$ -REPRESENTATION OF CCR

Standard position and momentum operators x, p with $[x, p] = i$ have the x - p rotation covariant property, that is,

$$x \cos \theta + p \sin \theta = e^{i\theta H} x e^{-i\theta H}$$

where H is given by $x^2 + p^2$. This property is essential when dealing with quantum harmonic oscillators. On the other hand, $Q^{(n)}$ and $P^{(m)}$ defined above have no clear covariant property of this sort. This fact reveals a defect of these matrices. In this section, we argue another hyperfinite-dimensional representation of CCR, which have the clear rotation covariance.

Let J_1, J_2 and J_3 be an irreducible set of $(p+1) \times (p+1)$ hermitian matrices which satisfies

$$[J_k, J_l] = i\epsilon_{klm} J_m$$

where ϵ_{klm} is Levi-Civita symbol. iJ_1, iJ_2 and iJ_3 generate a $p+1$ dimensional irreducible representation of Lie algebra $so(3)$. It is known that J_k has eigenvalues

$$\frac{p}{2}, \quad \frac{p}{2} - 1, \dots, \quad 1 - \frac{p}{2}, \quad -\frac{p}{2}$$

and corresponding eigenvectors are given by

$$|J_3; \frac{p}{2}\rangle, \quad J_-|J_3; \frac{p}{2}\rangle, \dots, \quad J_-^p|J_3; \frac{p}{2}\rangle$$

where $|J_3; \frac{p}{2}\rangle$ is a normalized eigenvector with eigenvalue $p/2$ and $J_- = J_1 - iJ_2$. Let $j = p/2$ and $|J_3; m\rangle = J_-^{j-m}|J_3; j\rangle / \|J_-^{j-m}|J_3; j\rangle\|$. It is shown that

$$\langle J_3; m' | J_- | J_3; m \rangle = \delta_{m' m-1} \sqrt{(j+m)(j-m+1)}$$

We consider the nonstandard version of this representation; let p be an infinite hypernatural number, and J_k be the internal matrices satisfying the above commutation relations. Define Q and P by $Q = J_1/\sqrt{j}$, $P = J_2/\sqrt{j}$.

Theorem 3.1: Let $S \subset {}^*\mathbf{C}^\nu$ be the subspace finitely spanned by $|J_3; j-k\rangle$, ($k = 0, 1, \dots < \infty$), i.e.,

$$S = \left\{ \sum_{k=0}^n c_k |J_3; j-k\rangle : c_k \in {}^*\mathbf{C}, |c_k| < \infty, n \in \mathbf{N} \right\}$$

S is an invariant subspace of P and Q , and

$$[Q, P]|\xi\rangle \approx i|\xi\rangle$$

for each $|\xi\rangle \in S$, and hence (Q, P, S) is a representation of CCR.

Proof: The invariance of S is seen from

$$\begin{aligned} Q|J_3; j-k\rangle &= \frac{1}{\sqrt{j}} \frac{1}{2} (J_- + J_-^\dagger) |J_3; j-k\rangle \\ &= \frac{1}{2} \sqrt{(2-k/j)(k+1)} |J_3; j-k-1\rangle + \frac{1}{2} \sqrt{k(2-k/j+1/j)} |J_3; j-k+1\rangle \in S \end{aligned}$$

$$\begin{aligned} P|J_3; j-k\rangle &= \frac{1}{\sqrt{j}} \frac{i}{2} (J_-^\dagger - J_-) |J_3; j-k\rangle \\ &= \frac{i}{2} \sqrt{(2-k/j)(k+1)} |J_3; j-k-1\rangle - \frac{i}{2} \sqrt{k(2-k/j+1/j)} |J_3; j-k+1\rangle \in S \end{aligned}$$

And we have $[Q, P]|J_3; j-k\rangle = \frac{i}{j} J_3 |J_3; j-k\rangle = i(1-k/j)|J_3; j-k\rangle \approx i|J_3; j-k\rangle$. Q.E.D.

We find the rotation covariance of these operators;

$$Q \cos \theta + P \sin \theta = e^{i\theta J_3} Q e^{-i\theta J_3}.$$

Moreover, it turns out that this representation is convenient in dealing with coherent states. Let $R(\theta, \phi)$ denote the rotation matrix

$$R(\theta, \phi) = e^{i\theta(J_1 \sin \phi - J_2 \cos \phi)}$$

$R(\theta, \phi)$ has also the expression¹¹

$$R(\theta, \phi) = e^{\mu J_-} e^{-\log(1+|\mu|^2)J_3} e^{-\mu^* J_-^\dagger}$$

where $\mu = e^{i\phi} \tan \frac{\theta}{2}$. Define $|\theta, \phi\rangle$ by $|\theta, \phi\rangle = R(\theta, \phi)|J_3; j\rangle$. It is shown that

$$|\theta, \phi\rangle = \frac{1}{(1+|\mu|^2)^j} \sum_{k=0}^{2j} \binom{2j}{k}^{1/2} \mu^k |J_3; j-k\rangle$$

If $|\mu|\sqrt{j} < \infty$ and $k < \infty$,

$$\frac{1}{(1+|\mu|^2)^j} \approx e^{-j|\mu|^2}, \quad \binom{2j}{k}^{1/2} \mu^k \approx \frac{(\sqrt{2j}\mu)^k}{\sqrt{k!}}$$

Thus we find

$$\langle J_3; j-k | \theta, \phi \rangle \approx e^{-|z|^2/2} \frac{z^k}{k!}$$

where $z = \mu\sqrt{2j}$. Corresponding standard coherent state $|z\rangle$ satisfies

$$\langle k | z \rangle = e^{-|z|^2/2} \frac{z^k}{k!}$$

where $|k\rangle = a^{\dagger k}|0\rangle/\|a^{\dagger k}|0\rangle\|$, a^\dagger being the standard bose creation operator and $|0\rangle$ the vacuum state.

IV. REPRESENTATIONS OF $SO(n)$

The hyperfinite-dimensional representation of CCR which has more general rotation covariance is constructed by hyperfinite-dimensional representation of $so(n)$. This section reviews the method of *spin representation* in use of Clifford algebra. Let $C(\mathbf{R}_-^n)$ be the \mathbf{R} -vector space the base of which is the 2^n elements

$$1, e_1, \dots, e_n, e_1 e_2, \dots, e_i e_j \ (i < j), \dots, e_{n-1} e_n, \dots,$$

$$e_{i_1} \cdots e_{i_k} \ (i_1 < \cdots < i_k), \dots, e_1 \cdots e_n$$

Define the product of these elements by

$$(e_{i_1} \cdots e_{i_k})(e_{j_1} \cdots e_{j_l}) = e_{i_1} \cdots e_{i_k} e_{j_1} \cdots e_{j_l}$$

and require the relation

$$e_i^2 = -1, \quad (i = 1, \dots, n), \quad e_i e_j = -e_j e_i \quad (i \neq j), \quad (\lambda 1) e_i = e_i (\lambda 1) \quad (\lambda \in \mathbf{R})$$

Assume that 1 is the unit of the product, and that the product is associative. Now, $C(\mathbf{R}_-^n)$ is 2^n -dimensional \mathbf{R} -algebra. $C(\mathbf{R}_-^n)$ is called *Clifford algebra*.

Let us identify $(\lambda_1, \dots, \lambda_n) \in \mathbf{R}^n$ with $\sum_{k=1}^n \lambda_k e_k$. Then we have $S^{n-1} \subset \mathbf{R}^n \subset C(\mathbf{R}_-^n)$ (S^{n-1} is the unit sphere in \mathbf{R}^n). The following fact is shown.

Theorem 4.1: *Spin*(n) defined by

$$Spin(n) = \{\alpha \in C(\mathbf{R}_-^n) \mid \alpha = a_1 \cdots a_m, a_i \in S^{n-1}, m = 2, 4, \dots\}$$

is a topological group, and isomorphic to the covering group of $SO(n)$. $e_{ij} = e_i e_j$ ($i, j = 1, \dots, n$) is a base (of a representation) of $so(n)$.

If $n = 2\nu + 1$, e_1, \dots, e_n is represented by $2^\nu \times 2^\nu$ matrices as follows. Let $V_k \simeq \mathbf{C}^2$ ($k = 1, \dots, \nu$). Pauli matrices $\sigma_{1,k}, \sigma_{2,k}, \sigma_{3,k}$ that act on V_k are represented as

$$\sigma_{1,k} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{2,k} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{3,k} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define V by

$$V = V_1 \otimes \cdots \otimes V_\nu,$$

and define $\hat{\sigma}_{c,k}$ and γ_i that acts on V by

$$\hat{\sigma}_{c,k} = \overbrace{1 \otimes \cdots \otimes 1}^{k-1} \otimes \sigma_{c,k} \otimes 1 \otimes \cdots \otimes 1, \quad c = 1, 2, 3,$$

$$\gamma_{2k-1} = \hat{\sigma}_{2,k} \hat{\sigma}_{3,k+1} \cdots \hat{\sigma}_{3,\nu},$$

$$\gamma_{2k} = -\hat{\sigma}_{1,k} \hat{\sigma}_{3,k+1} \cdots \hat{\sigma}_{3,\nu},$$

$$\gamma_{2\nu+1} = \hat{\sigma}_{3,1} \hat{\sigma}_{3,2} \cdots \hat{\sigma}_{3,\nu}.$$

Direct calculations show the following relations.

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}, \quad i, j = 1, \dots, 2\nu.$$

Thus, we can take $e_j = i\gamma_j$ (this is called the *spin representation*).

Moreover we have the tensor product representation of $so(n)$ from one constructed above;

$$\tilde{e}_{ij} = \sum_{l=0}^{p-1} \overbrace{1 \otimes \cdots \otimes 1}^l \otimes e_{ij} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{p-l-1}$$

also generate a representation of $so(n)$.

In the next section, we introduce a hyperfinite representation of $so(n)$, using these mathematical tools.

V. HYPERFINITE PARA-FERMI REPRESENTATION

In this section, we construct a hyperfinite-dimensional representation of the algebra of standard infinite bose oscillators, that is, the algebra generated by bose annihilation operators a_1, a_2, \dots satisfying $[a_j, a_k^\dagger] = \delta_{jk}$ and $[a_j, a_k] = 0$, with the condition that nonzero vector $|0\rangle$ that satisfies $a_k|0\rangle = 0$ is unique except the scalar multiples (*the uniqueness of vacuum*). This representation is constructed by using hyperfinite internal representations of the algebra of parafermi oscillators, which is mathematically equivalent to the hyperfinite-dimensional spin representation of $so(n)$.

Definition 5.1:^{12,13} Let $\nu \in \mathbf{N}$, and suppose that for some $d \in \mathbf{N}$, $b_1, \dots, b_\nu \in M(d, \mathbf{C})$ (i.e., b_1, \dots, b_ν are finite-dimensional matrices). b_1, \dots, b_ν is called *annihilation operators* of parafermi oscillators of order $p \in \mathbf{N}$ if they satisfy

$$\begin{aligned} [b_k, [b_l^\dagger, b_m]] &= 2\delta_{kl}b_m \\ [b_k, [b_l^\dagger, b_m^\dagger]] &= 2\delta_{kl}b_m^\dagger - 2\delta_{km}b_l^\dagger \\ [b_k, [b_l, b_m]] &= 0. \end{aligned}$$

and the uniqueness of vacuum $|0\rangle$, and,

$$b_k b_l^\dagger |0\rangle = \delta_{kl} p |0\rangle.$$

$b_1^\dagger, \dots, b_\nu^\dagger$ are called *creation operators* of parafermi oscillators of order p . *Hyperfinite annihilation operators* of parafermi oscillators are the internal matrices defined in the above definition by substituting ${}^*\mathbf{N}$ and ${}^*\mathbf{C}$ for \mathbf{N} and \mathbf{C} , respectively.

Green¹² has given a class of representations of the above commutation relations of parafermi creation and annihilation operators. In the so-called *Green representation* for the cases of order p , the parafermi operators b_k are expressed by the form

$$b_k = \sum_{\alpha=1}^p b_k^{(\alpha)},$$

where the *Green-component* operators $b_k^{(\alpha)}$ satisfy the commutation relations

$$\{b_k^{(\alpha)}, b_l^{(\alpha)\dagger}\} = \delta_{kl}, \quad \{b_k^{(\alpha)}, b_l^{(\alpha)}\} = 0,$$

$$[b_k^{(\alpha)}, b_l^{(\beta)\dagger}] = [b_k^{(\alpha)}, b_l^{(\beta)}] = 0, \quad (\alpha \neq \beta)$$

where $\{A, B\} = AB + BA$, and the uniqueness of vacuum $|0\rangle$ such that

$$b_k^{(\alpha)} |0\rangle = 0 \quad \text{for all } k, \alpha.$$

Green representation is essentially equivalent to the tensor product representation of Clifford algebra representation of $so(2\nu)$. In fact, we easily verify that $e_1, \dots, e_{2\nu}$ defined by $e_{2k-1} = i(b_k^\dagger + b_k)$, $e_{2k} = b_k^\dagger - b_k$ form the generator of a Clifford algebra (i.e., $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ ($i \neq j$) hold). Thus, we can construct a $2^{p\nu}$ -dimensional representation of Green components by using a spin representation of the Clifford algebra as follows. Let $V_k^{(\alpha)} \simeq \mathbf{C}^2$ ($k = 1, \dots, \nu$, $\alpha = 1, \dots, p$). Pauli matrices $\sigma_{1,k}^{(\alpha)}$, $\sigma_{2,k}^{(\alpha)}$ and $\sigma_{3,k}^{(\alpha)}$ that act on $V_k^{(\alpha)}$ are represented as

$$\sigma_{1,k}^{(\alpha)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_{2,k}^{(\alpha)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_{3,k}^{(\alpha)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define $V^{(\alpha)}$ by

$$V^{(\alpha)} = V_1^{(\alpha)} \otimes \dots \otimes V_\nu^{(\alpha)},$$

and define $\hat{\sigma}_{c,k}^{(\alpha)}$ and $\gamma_i^{(\alpha)}$ ($k = 1, \dots, \nu$, $i = 1, \dots, 2\nu$) that acts on $V^{(\alpha)}$ by

$$\hat{\sigma}_{c,k}^{(\alpha)} = \overbrace{1 \otimes \dots \otimes 1}^{k-1} \otimes \sigma_{c,k}^{(\alpha)} \otimes 1 \otimes \dots \otimes 1, \quad c = 1, 2, 3,$$

$$\gamma_{2k-1}^{(\alpha)} = \hat{\sigma}_{2,k}^{(\alpha)} \hat{\sigma}_{3,k+1}^{(\alpha)} \dots \hat{\sigma}_{3,\nu}^{(\alpha)},$$

$$\gamma_{2k}^{(\alpha)} = -\hat{\sigma}_{1,k}^{(\alpha)} \hat{\sigma}_{3,k+1}^{(\alpha)} \dots \hat{\sigma}_{3,\nu}^{(\alpha)}.$$

Operators $b_k^{(\alpha)}$ ($k = 1, \dots, \nu$) defined by

$$b_k^{(\alpha)} = \frac{1}{2}(\gamma_{2k-1}^{(\alpha)} - i\gamma_{2k}^{(\alpha)})$$

satisfy the relations

$$\{b_k^{(\alpha)}, b_l^{(\alpha)\dagger}\} = \delta_{kl}, \quad \{b_k^{(\alpha)}, b_l^{(\alpha)}\} = 0,$$

for all $k, l = 1, \dots, \nu$.

Define V and $\tilde{b}_k^{(\alpha)}$ ($k = 1, \dots, \nu$) acting on V by

$$V = V^{(1)} \otimes \dots \otimes V^{(p)},$$

$$\tilde{b}_k^{(\alpha)} = \overbrace{1 \otimes \dots \otimes 1}^{\alpha-1} \otimes b_k^{(\alpha)} \otimes 1 \otimes \dots \otimes 1.$$

We see that for all $k, l = 1, \dots, \nu$,

$$\{\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\alpha)\dagger}\} = \delta_{kl}, \quad \{\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\alpha)}\} = 0,$$

$$[\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\beta)\dagger}] = [\tilde{b}_k^{(\alpha)}, \tilde{b}_l^{(\beta)}] = 0, \quad (\alpha \neq \beta).$$

Let $|0\rangle_k^{(\alpha)} \in V_k^{(\alpha)}$ denote the normalized vector such that $b_k^{(\alpha)}|0\rangle_k^{(\alpha)} = 0$. Define $|0\rangle^{(\alpha)}$ and $|0\rangle$ by

$$\begin{aligned} |0\rangle^{(\alpha)} &= |0\rangle_1^{(\alpha)} \otimes \cdots \otimes |0\rangle_\nu^{(\alpha)}, \\ |0\rangle &= |0\rangle^{(\alpha)} \otimes \cdots \otimes |0\rangle^{(p)}. \end{aligned}$$

Now, we find that $\tilde{b}_1^{(\alpha)}, \dots, \tilde{b}_\nu^{(\alpha)}$ are $2^{p\nu}$ -dimensional representations of Green components and $|0\rangle$ is the vacuum. Thus, $b_k = \sum_{\alpha=1}^p \tilde{b}_k^{(\alpha)}$, ($k = 1, \dots, \nu$) are $2^{p\nu}$ -dimensional representations of annihilation operators of ν parafermi oscillators of order p . Let us call the above representation of the algebra of parafermi oscillators *spin representation*.

Define $\sigma_{\pm,k}^{(\alpha)}$ by

$$\sigma_{\pm,k}^{(\alpha)} = (\sigma_{1,k}^{(\alpha)} \pm i\sigma_{2,k}^{(\alpha)})/2.$$

and $|1\rangle_k^{(\alpha)} \in V_k^{(\alpha)}$ by

$$|1\rangle_k^{(\alpha)} = \sigma_{+,k}^{(\alpha)} |0\rangle_k^{(\alpha)}.$$

$$\{(|e_1^{(1)}\rangle_1^{(1)} \cdots |e_\nu^{(1)}\rangle_\nu^{(1)}) \cdots (|e_1^{(p)}\rangle_1^{(p)} \cdots |e_\nu^{(p)}\rangle_\nu^{(p)}) : e_k^{(\alpha)} = 0, 1\}$$

(\otimes 's are omitted) is a complete orthonormal system of V . We write the vectors simply as $|\{e_k^{(\alpha)}\}\rangle$.

Number operator N on V and the related operators $N_k, N^{(\alpha)}$ are defined as follows:

$$\begin{aligned} N_k^{(\alpha)} &= \frac{1}{2}(1 + \sigma_{3,k}^{(\alpha)}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ \hat{N}_k^{(\alpha)} &= \overbrace{1 \otimes \cdots \otimes 1}^{k-1} \otimes N_k^{(\alpha)} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{\nu-k}, \\ \tilde{N}_k^{(\alpha)} &= \overbrace{1 \otimes \cdots \otimes 1}^{\alpha-1} \otimes \hat{N}_k^{(\alpha)} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{p-\alpha}, \\ N_k &= \sum_{\alpha=1}^p \tilde{N}_k^{(\alpha)}, \quad N^{(\alpha)} = \sum_{k=1}^\nu \tilde{N}_k^{(\alpha)}, \quad N = \sum_{\alpha=1}^p N^{(\alpha)}, \end{aligned}$$

We see that

$$N|\{e_k^{(\alpha)}\}\rangle = n|\{e_k^{(\alpha)}\}\rangle$$

where n is the number of $e_k^{(\alpha)}$'s that is equal to 1. It is easily shown that

$$\tilde{b}_k^{(\alpha)\dagger} \tilde{b}_k^{(\alpha)} = \tilde{N}_k^{(\alpha)}, \quad \tilde{b}_k^{(\alpha)} \tilde{b}_k^{(\alpha)\dagger} = 1 - \tilde{N}_k^{(\alpha)}, \quad N_k = \frac{1}{2}([b_k^\dagger, b_k] + p),$$

$$[N_k, N_l] = 0, \quad N_k b_k = b_k(N_k - 1), \quad N_k b_k^\dagger = b_k^\dagger(N_k + 1), \quad \text{etc.}$$

Lemma 5.1: Suppose that hyperfinite parafermi annihilation operators b_1, \dots, b_ν are represented by spin representation, and that their order p is an infinite hypernatural number (b_1, \dots, b_ν are $2^{p\nu} \times 2^{p\nu}$ internal matrices acting on ${}^*\mathbf{C}^{2^{p\nu}}$). If $|\xi\rangle \in {}^*\mathbf{C}^{2^{p\nu}}$ satisfies $\langle \xi | \xi \rangle, \langle \xi | N^2 | \xi \rangle < \infty$, and $k \neq l$ ($k, l = 1, 2, \dots, \nu$), then

- (i) $[\beta_k, \beta_l]|\xi\rangle \approx [\beta_k, \beta_l^\dagger]|\xi\rangle \approx 0$.
- (ii) $[\beta_k, \beta_k^\dagger]|\xi\rangle \approx |\xi\rangle$.
- (iii) $\beta_k \beta_k^{\dagger n} |\xi\rangle \approx (\beta_k^{\dagger n} \beta_k + n \beta_k^{\dagger n-1})|\xi\rangle$.

where $\beta_k = p^{-1/2} b_k$ (*normalization* of b_k) and $n < \infty$.

Proof: (i) Notice that

$$[\beta_k, \beta_l] = \frac{2}{p} \sum_{\alpha=1}^p \tilde{b}_k^{(\alpha)} \tilde{b}_l^{(\alpha)}, \quad [\beta_k, \beta_l^\dagger] = \frac{2}{p} \sum_{\alpha=1}^p \tilde{b}_k^{(\alpha)} \tilde{b}_l^{(\alpha)\dagger}.$$

Direct calculations show that

$$b_k^{(\alpha)} b_l^{(\alpha)} = \begin{cases} -\hat{\sigma}_{3,k+1}^{(\alpha)} \cdots \hat{\sigma}_{3,l}^{(\alpha)} \hat{\sigma}_{-,k}^{(\alpha)} \hat{\sigma}_{-,l}^{(\alpha)} & (l > k) \\ -\hat{\sigma}_{3,l+1}^{(\alpha)} \cdots \hat{\sigma}_{3,k}^{(\alpha)} \hat{\sigma}_{-,k}^{(\alpha)} \hat{\sigma}_{-,l}^{(\alpha)} & (k > l) \end{cases}$$

$$b_k^{(\alpha)} b_l^{(\alpha)\dagger} = \begin{cases} \hat{\sigma}_{3,k+1}^{(\alpha)} \cdots \hat{\sigma}_{3,l}^{(\alpha)} \hat{\sigma}_{+,k}^{(\alpha)} \hat{\sigma}_{-,l}^{(\alpha)} & (l > k) \\ \hat{\sigma}_{3,l+1}^{(\alpha)} \cdots \hat{\sigma}_{3,k}^{(\alpha)} \hat{\sigma}_{+,k}^{(\alpha)} \hat{\sigma}_{-,l}^{(\alpha)} & (k > l) \end{cases}$$

Hence, using $N_k^{(\alpha)} + N_l^{(\alpha)} - 2N_k^{(\alpha)} N_l^{(\alpha)} = (N_k^{(\alpha)} - N_l^{(\alpha)})^2 \geq 0$,

$$\begin{aligned} \|\tilde{b}_k^{(\alpha)} \tilde{b}_l^{(\alpha)} |\xi\rangle\|^2 &= \langle \xi | \tilde{\sigma}_{+,k}^{(\alpha)} \tilde{\sigma}_{+,l}^{(\alpha)} \tilde{\sigma}_{-,l}^{(\alpha)} \tilde{\sigma}_{-,k}^{(\alpha)} | \xi \rangle \\ &= \langle \xi | \tilde{N}_k^{(\alpha)} \tilde{N}_l^{(\alpha)} | \xi \rangle \\ &\leq \frac{1}{2} \langle \xi | \tilde{N}_k^{(\alpha)} + \tilde{N}_l^{(\alpha)} | \xi \rangle \\ &\leq \frac{1}{2} \langle \xi | \tilde{N}^{(\alpha)} | \xi \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} \|[\beta_k, \beta_l]|\xi\rangle\| &= \left\| \frac{2}{p} \sum_{\alpha=1}^p \tilde{b}_k^{(\alpha)} \tilde{b}_l^{(\alpha)} |\xi\rangle \right\| \\ &\leq \frac{2}{p} \sum_{\alpha=1}^p \|\tilde{b}_k^{(\alpha)} \tilde{b}_l^{(\alpha)} |\xi\rangle\| \\ &\leq \frac{2}{p} \sum_{\alpha=1}^p \sqrt{\frac{1}{2} \langle \xi | \tilde{N}^{(\alpha)} | \xi \rangle} \approx 0. \end{aligned}$$

The last equation is seen from $\sum_{\alpha=1}^p \langle \xi | \tilde{N}^{(\alpha)} | \xi \rangle = \langle \xi | \tilde{N} | \xi \rangle < \infty$; under this condition we find that the number of α 's such that $\langle \xi | \tilde{N}^{(\alpha)} | \xi \rangle \not\approx 0$ is finite.

In a similar way, it is shown that

$$\|[\beta_k, \beta_l^\dagger]|\xi\rangle\| \leq \frac{2}{p} \sum_{\alpha=1}^p \|\tilde{b}_k^{(\alpha)} \tilde{b}_l^{(\alpha)\dagger} |\xi\rangle\|$$

$$\begin{aligned}
&\leq \frac{2}{p} \sum_{\alpha=1}^p \sqrt{\langle \xi | \tilde{N}_l^{(\alpha)} (1 - \tilde{N}_k^{(\alpha)}) | \xi \rangle} \\
&\leq \frac{2}{p} \sum_{\alpha=1}^p \sqrt{\langle \xi | \tilde{N}_l^{(\alpha)} | \xi \rangle} \approx 0.
\end{aligned}$$

(ii) A direct calculation shows that

$$\begin{aligned}
[\beta_k, \beta_k^\dagger] &= -\frac{1}{p} \sum_{\alpha=1}^p \tilde{\sigma}_{3,k}^{(\alpha)}, \\
([\beta_k, \beta_k^\dagger] - 1)^2 &= \frac{4}{p^2} \left(\sum_{\alpha=1}^p N_k^{(\alpha)} \right)^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|[\beta_k, \beta_k^\dagger]|\xi\rangle - |\xi\rangle\|^2 &= \frac{4}{p^2} \langle \xi | \left(\sum_{\alpha=1}^p N_k^{(\alpha)} \right)^2 | \xi \rangle. \\
&\leq \frac{4}{p^2} \langle \xi | N^2 | \xi \rangle \approx 0.
\end{aligned}$$

(iii) is shown by the induction for n , using (ii). Q.E.D.

Suppose the number of parafermi oscillators ν and their order p are infinite hypernatural numbers. When n_i is nonnegative integer for any $i = 1, 2, \dots < \infty$, and the number of n_i 's such that $n_i \neq 0$ is finite, we will define $|n_1, n_2, \dots\rangle$ by

$$|n_1, n_2, \dots\rangle = \frac{b_1^{\dagger n_1} b_2^{\dagger n_2} \dots |0\rangle}{\|b_1^{\dagger n_1} b_2^{\dagger n_2} \dots |0\rangle\|}.$$

$b_1^{\dagger n_1} b_2^{\dagger n_2} \dots$ is the product of a finite number of operators, so it is well-defined. $N_k |n_1, n_2, \dots\rangle = n_k |n_1, n_2, \dots\rangle$ is easily shown, and hence, since N_k is hermitian, the set of the vectors of the form $|n_1, n_2, \dots\rangle$ is an orthonormal system.

Lemma 5.2:

- (i) $\beta_k^\dagger \beta_k |n_1, n_2, \dots\rangle \approx n_k |n_1, n_2, \dots\rangle,$
- (ii) $\beta_k \beta_k^\dagger |n_1, n_2, \dots\rangle \approx (n_k + 1) |n_1, n_2, \dots\rangle,$
- (iii) $\|\beta_1^{\dagger n_1} \beta_2^{\dagger n_2} \dots |0\rangle\| \approx \sqrt{n_1! n_2! \dots},$
- (iv) $\beta_k^\dagger |n_1, n_2, \dots\rangle \approx \sqrt{n_k + 1} |n_1, n_2, \dots, n_k + 1, \dots\rangle,$
- (v) $\beta_k |n_1, n_2, \dots\rangle \approx \sqrt{n_k} |n_1, n_2, \dots, n_k - 1, \dots\rangle.$

Proof: (i) and (ii) are shown by Lemma 5.1(i)-(iii). (iii) is shown by the induction for n_1, n_2, \dots , using (ii) and Lemma 5.1(i). (iv) and (v) are shown by (iii) and Lemma 5.1(i)(iii). Q.E.D.

Define a set $D \subset {}^*\mathbf{C}^{2^{p\nu}}$ by

$$D = \left\{ \frac{\beta_{k_1}^\dagger \cdots \beta_{k_n}^\dagger |0\rangle}{\|\beta_{k_1}^\dagger \cdots \beta_{k_n}^\dagger |0\rangle\|} \mid n, k_1, \dots, k_n \in \mathbf{N} \right\} \cup \{|0\rangle\}.$$

Clearly, every vector in D is a normalized eigenvector of the number operator N with a finite eigenvalue. Let S denote the external subspace of ${}^*\mathbf{C}^{2^{p\nu}}$ spanned by D , i.e.,

$$S = \left\{ \sum_{i=1}^n c_i |\xi\rangle : c_i \in {}^*\mathbf{C}, |c_i| < \infty, n \in \mathbf{N}, |\xi\rangle \in D \right\}.$$

The following theorem follows from Lemma 5.1 and 5.2.

Theorem 5.3: (β_k, S) ($k \in \mathbf{N}$) is a representation of CCR of countably-infinite degree of freedom, i.e., S is invariant in respect to β_k and β_k^\dagger for every $k \in \mathbf{N}$, and

$$\begin{aligned} [\beta_k, \beta_l] |\xi\rangle &\approx 0 \\ [\beta_k, \beta_l^\dagger] |\xi\rangle &\approx \delta_{kl} |\xi\rangle \end{aligned}$$

for any $|\xi\rangle \in S$. Moreover, the uniqueness of vacuum is satisfied in the following sense: if $|\xi\rangle \in S$, $\langle \xi | \xi \rangle = 1$ and $\beta_k |\xi\rangle \approx 0$ for all $k \in \mathbf{N}$, then $|\langle \xi | 0 \rangle| \approx 1$.

APPENDIX: NONSTANDARD ANALYSIS

This section briefly outlines the basic notions of nonstandard analysis along Hurd and Loeb⁶, and reviews some development in the field of hyperfinite-dimensional linear space theory. Let X be a set and $\mathcal{P}(X)$ the power set of X , that is, the set of all subsets of X . The *superstructure over X* , denoted by $V(X)$, is defined by the following recursion:

$$V_0(X) = X, V_{n+1}(X) = V_n(X) \cup \mathcal{P}(V_n(X)),$$

$$V(X) = \bigcup_{n \in \mathbf{N}} V_n(X).$$

Let us regard any element of X as a nonset here; hence $x \in V(X)$ is a set iff $x \in V(X) \setminus X$. Let \mathbf{C} be the set of complex numbers. $V(\mathbf{C})$ contains all the structures that we usually use in quantum physics, for instance, separable Hilbert space \mathcal{H} .

$V(X)$ is called a *nonstandard extension of $V(\mathbf{C})$* if there exists a map $\star : V(\mathbf{C}) \longrightarrow V(X)$ satisfying the following conditions:

- (1) \star is an injective mapping from $V(\mathbf{C})$ to $V(X)$,

$$(2) {}^*\mathbf{C} = X,$$

(3) (Transfer Principle) Let ϕ be a sentence in terms of $V(\mathbf{C})$, and ${}^*\phi$ the sentence “transferred” from ϕ by mapping \star . ϕ is true iff ${}^*\phi$ is true.

Transfer Principle needs more explanation. Although the exact description of it needs more definitions of some notions in mathematical logic, one can find the intuitive meaning of it without them, in an example. A sentence in terms of $V(\mathbf{C})$ is constructed from the symbols for logical connectives $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$, quantifiers \forall, \exists , individual variables x, y, z, \dots , two predicates $=, \in$, parentheses $(,)$, and elements of $V(\mathbf{C})$. We will consider an example. Let \mathbf{R} denote the set of real numbers. Define $G_< \in V(\mathbf{C})$ by $G_< = \{(x, y) \mid x, y \in \mathbf{R}, x < y\}$, where (x, y) is identified as $\{\{x\}, y\}$.

$$(\forall x)(\forall y)(x \in \mathbf{R} \wedge y \in \mathbf{R} \wedge (x, y) \in G_< \Rightarrow (\exists z)(z \in \mathbf{R}$$

$$\wedge (x, z) \in G_< \wedge (z, y) \in G_<))$$

is a sentence in terms of $V(\mathbf{C})$ because $\mathbf{R}, G_< \in V(\mathbf{C})$. Let ϕ denote this sentence. ϕ means that \mathbf{R} is dense, and hence ϕ is true. The “transferred” sentence ${}^*\phi$ is as follows:

$$(\forall x)(\forall y)(x \in {}^*\mathbf{R} \wedge y \in {}^*\mathbf{R} \wedge (x, y) \in {}^*G_< \Rightarrow (\exists z)(z \in {}^*\mathbf{R}$$

$$\wedge (x, z) \in {}^*G_< \wedge (z, y) \in {}^*G_<))$$

By Transfer Principle, ${}^*\phi$ is true (${}^*\mathbf{R}$ is called \star -dense).

$u \in V(X)$ is called *standard* if there is $x \in V(\mathbf{C})$ such that $u = {}^*x$, and called *internal* if there is $x \in V(\mathbf{C})$ such that $u \in {}^*x$. ${}^*V(\mathbf{C})$ is the set of all internal sets. Let A and B be internal sets. Function $f : A \longrightarrow B$ is called *internal* if the graph of f (i.e., $\{(x, f(x)) \mid x \in A\}$) is internal. $V(X)$ is called a *countably saturated extension* of $V(\mathbf{C})$ if it satisfies the following condition:

(Saturation Principle) If countable sequence of internal sets $A_j \in V(X) \setminus X$ satisfies

$$\bigcap_{j=1}^k A_j \neq \emptyset \quad (k = 1, 2, \dots)$$

then

$$\bigcap_{j=1}^{\infty} A_j \neq \emptyset.$$

Let $A \in V(\mathbf{C}) \setminus \mathbf{C}$. From the Saturation Principle, we can show that if A is an infinite set, sA defined by

$${}^sA = \{{}^*a \mid a \in A\}$$

is a proper subset of *A . Hence *A is an extended structure of A . *A is called the *nonstandard extension* of A . By renaming the elements of X , we can assume without loss of generality that \mathbf{C} is a subset of ${}^*\mathbf{C}$ and ${}^*x = x$ for each $x \in \mathbf{C}$.

Given any subset $U \subseteq V(\mathbf{C})$, define *U by

$${}^*U = \bigcup_{n \in \mathbf{N}} {}^*(U \cap V_n(\mathbf{C})).$$

Let $F(\mathbf{C}) \subseteq V(\mathbf{C})$ be the set of finite sets, i.e.,

$$F(\mathbf{C}) = \{A \in V(\mathbf{C}) \setminus \mathbf{C} \mid A \text{ is a finite set}\}.$$

An element of ${}^*F(\mathbf{C})$ is called a *hyperfinite set*. It is shown that if A is a hyperfinite set, then there is an initial segment $J = \{n \in {}^*\mathbf{N} \mid n \leq j\}$ for some $j \in {}^*\mathbf{N}$ and a one-to-one, onto internal mapping $f : J \rightarrow A$. Thus, we will often write a hyperfinite set A as $A = \{a_1, a_2, \dots, a_j\}$, where $a_k = f(k)$, $k \in J$.

Any element of ${}^*\mathbf{C}({}^*\mathbf{R})$ is called a *hypercomplex (hyperreal) number*. We assume that ${}^*\mathbf{R} \subseteq {}^*\mathbf{C}$. ${}^*\mathbf{C}$ is a proper extension field of \mathbf{C} , and ${}^*\mathbf{R}$ is an ordered extension of \mathbf{R} . An element of ${}^*\mathbf{N}({}^*\mathbf{Z})$ is called a *hypernatural number (hyperinteger)*. A hypercomplex number x is called *infinite* if $|x| > n$ for any $n \in \mathbf{N}$, *finite*, $|x| < \infty$, if there is some $n \in \mathbf{N}$ such that $|x| < n$, and *infinitesimal* if $|x| < \frac{1}{n}$ for any $n \in \mathbf{N}$.

For any $x, y \in {}^*\mathbf{C}$, we will write $x \approx y$ if $|x - y|$ is infinitesimal. For any finite hypercomplex number x , there is a unique complex number z such that ${}^*z \approx x$; this z is called the *standard part of* x and denoted by ${}^\circ x$.

Let \mathcal{A} be an internal normed linear space with norm $\|\cdot\|$. The *principal galaxy* $\text{fin}(\mathcal{A})$ and the *principal monad* $\mu(0)$ are defined by

$$\text{fin}(\mathcal{A}) = \{x \in \mathcal{A} \mid \|x\| < \infty\}$$

$$\mu(0) = \{x \in \mathcal{A} \mid \|x\| \approx 0\}$$

Both of them are linear spaces over \mathbf{C} . The *nonstandard hull* of \mathcal{A} is the quotient linear space $\hat{\mathcal{A}} = \text{fin}(\mathcal{A})/\mu(0)$ equipped with the norm given by

$$\|{}^\circ x\| = {}^\circ \|x\|$$

for all $x \in \text{fin}(\mathcal{A})$, where ${}^\circ x = x + \mu(0)$. It is shown by the Saturation Principle that $\hat{\mathcal{A}}$ is a Banach space⁶.

Let ν be an infinite hypernatural number, i.e., $\nu \in {}^*\mathbf{N} \setminus \mathbf{N}$. Define ${}^*\mathbf{C}^\nu$ by

$${}^*\mathbf{C}^\nu = \{f \mid f : \{1, \dots, \nu\} \rightarrow {}^*\mathbf{C}; \text{ internal}\}.$$

${}^*\mathbf{C}^\nu$ is a ν -dimensional internal inner product space with the natural inner product and the internal norm $\|\cdot\|$ derived by the inner product. Define $\mathbf{M} = M({}^*\mathbf{C}, \nu)$, the set of $\nu \times \nu$ internal matrices over ${}^*\mathbf{C}$, by

$$\mathbf{M} = \{f \mid f : \{1, \dots, \nu\}^2 \rightarrow {}^*\mathbf{C}; \text{ internal}\}.$$

Naturally \mathbf{M} acts on ${}^*\mathbf{C}^\nu$ as the internal linear operators. Let p_∞ be the operator norm on \mathbf{M} , i.e., $p_\infty(A) = \sup\{\|A\xi\| \mid \|\xi\| \leq 1, \xi \in {}^*\mathbf{C}^\nu\}$. Denote by A^* the adjoint of $A \in \mathbf{M}$. Let τ be the internal normalized trace on \mathbf{M} , i.e.,

$$\tau(A) = \frac{1}{\nu} \sum_{i=1}^{\nu} A_{ii}$$

for $A = (A_{ij}) \in \mathbf{M}$. Then τ defines an internal inner product $(\cdot|\cdot)$ on \mathbf{M} by $(A|B) = \tau(A^*B)$, for $A, B \in \mathbf{M}$. Its derived norm called the normalized Hilbert-Schmidt norm is denoted by p_2 , i.e., $p_2(A) = \tau(A^*A)^{1/2}$ for $A \in \mathbf{M}$. Denote by (\mathbf{M}, p_∞) and (\mathbf{M}, p_2) the normed linear spaces equipped with these respective norms. The principal galaxies $\text{fin}_\infty(\mathbf{M})$ of (\mathbf{M}, p_∞) and $\text{fin}_2(\mathbf{M})$ of (\mathbf{M}, p_2) are defined as follows:

$$\text{fin}_\infty(\mathbf{M}) = \{A \in \mathbf{M} \mid p_\infty(A) < \infty\}$$

$$\text{fin}_2(\mathbf{M}) = \{A \in \mathbf{M} \mid p_2(A) < \infty\}$$

The principal monads $\mu_\infty(0)$ of (\mathbf{M}, p_∞) and $\mu_2(0)$ of (\mathbf{M}, p_2) are defined as follows:

$$\mu_\infty(0) = \{A \in \mathbf{M} \mid p_\infty(A) \approx 0\}$$

$$\mu_2(0) = \{A \in \mathbf{M} \mid p_2(A) \approx 0\}.$$

The nonstandard hull $\hat{\mathbf{M}}_2 = \text{fin}_2(\mathbf{M})/\mu_2(0)$ turns out to be a Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and norm $\|\cdot\|_2$ defined by

$$\langle A + \mu_2(0) \mid B + \mu_2(0) \rangle = {}^\circ(A|B)$$

and

$$\|A + \mu_2(0)\|_2 = p_2(A).$$

for $A, B \in \text{fin}_2(\mathbf{M})$.

The nonstandard hull $\hat{\mathbf{M}}_\infty = \text{fin}_\infty(\mathbf{M})/\mu_\infty(0)$ of (\mathbf{M}, p_∞) turns out to be a C^* -algebra equipped with norm \hat{p}_∞ defined by

$$\hat{p}_\infty(A + \mu_\infty(0)) = {}^\circ p_\infty(A)$$

for $A \in \text{fin}_\infty(\mathbf{M})$.

Hinokuma and Ozawa¹⁴ showed that another quotient space $\hat{\mathbf{M}}$ defined by

$$\hat{\mathbf{M}} = \text{fin}_\infty(\mathbf{M})/(\mu_2(0) \cap \text{fin}_\infty(\mathbf{M})).$$

is a von Neumann algebra of type II₁ factor.

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